In certain population models, a group will go extinct if and only if its population is below a certain level (called the survival threshold P_S).

SCORE: _____ / 12 PTS

NOTE: The answers of all parts of this question are likely to contain symbolic constants, such as P_S.

[a] Write a DE for a population of a group which is going extinct, if the rate of change of its population is proportional to the difference between the threshold and the existing population. **Write your DE so that all constants in your equation are POSITIVE.**
NOTE: This part of the question should be familiar.

ALL ITEMS 1 POINT UNLESS OTHERWISE NOTED

$$\underline{\frac{dP}{dt} = -k(P_s - P) \quad \text{OR} \quad k(P - P_s)}$$

[b] Assume that the population is initially P_0. Find a function which gives the population at time t.

$$\underline{\int -\frac{dP}{P_s - P} = \int k\,dt}$$

$$\underline{\ln|P_s - P| = kt + C}$$

$$P_s - P = Ce^{kt}$$

$$\underline{P = P_s + Ce^{kt}}$$

$$P_0 = P_s + C$$

$$C = P_0 - P_s \quad 2$$

$$\boxed{P(t) = P_s + (P_0 - P_s)e^{kt} \quad \text{OR } P_s - (P_s - P_0)e^{kt}}$$

[c] If the initial population is 90% of the survival threshold, determine when the population goes extinct.

$$P(t) = P_s - (P_s - 0.9P_s)e^{kt} = 0$$

$$\underline{P_s - 0.1P_s e^{kt} = 0} \quad 1\tfrac{1}{2}$$

$$e^{kt} = 10$$

$$t = \underline{\tfrac{1}{k}\ln 10} \quad 1\tfrac{1}{2}$$

[d] Notice that your work in part [b] did not algebraically require that P_0 be less than P_S. Analyze your final answer to part [b] to determine if this model seems appropriate if the initial population is greater than P_S. Justify your conclusion briefly.

IF $P_0 > P_s$, $\underline{P_0 - P_s > 0}$ AND $\underline{P(t) \to \infty \text{ AS } t \to \infty}$

THIS SEEMS INAPPROPRIATE IN THE SAME WAY THE ORIGINAL EXPONENTIAL MODEL SEEMED INAPPROPRIATE. - NO UPPER BOUND ON POPULATION

ALL ITEMS 1 POINT UNLESS OTHERWISE NOTED

A 1000 liter tank initially holds 500 liters of brine containing 2 grams of salt per liter. **SCORE: _____ / 18 PTS**
Brine containing 6 grams of salt per liter starts flowing into the tank at 15 liters per minute.
At the same time, the well-mixed solution leaves the tank at 5 liters per minute.

Find the amount of salt in the tank at the instant the tank overflows. **HINT: Simplify all fractions as soon as possible.**

$A(t)$ = AMOUNT OF SALT IN TANK @ TIME t MINUTES

$$\frac{dA}{dt} = 15\frac{L}{min}\cdot 6\frac{g}{L} - 5\frac{L}{min}\cdot\frac{A(t)\,g}{(500+(15-5)t)L}$$

$$\boxed{\frac{dA}{dt} = 90 - \frac{5A}{500+10t} = 90 - \frac{A}{100+2t}} \quad 6$$

$$\boxed{\frac{dA}{dt} + \frac{A}{100+2t} = 90}$$

$$\mu = e^{\int \frac{1}{100+2t}dt} = e^{\frac{1}{2}\ln|100+2t|} = \boxed{(100+2t)^{\frac{1}{2}}}$$

$$\boxed{(100+2t)^{\frac{1}{2}}\frac{dA}{dt} + (100+2t)^{-\frac{1}{2}}A = 90(100+2t)^{\frac{1}{2}}}$$

CHECK: $\boxed{\frac{d}{dt}(100+2t)^{\frac{1}{2}} = \frac{1}{\cancel{2}}(100+2t)^{-\frac{1}{2}}\cdot\cancel{2} = (100+2t)^{-\frac{1}{2}}}$ ✓

2 $$\boxed{(100+2t)^{\frac{1}{2}}A = \int 90(100+2t)^{\frac{1}{2}}dt = \overset{30}{\cancel{90}}\cdot\frac{2}{\cancel{3}}(100+2t)^{\frac{3}{2}}\cdot\frac{1}{2} + C = 30(100+2t)^{\frac{3}{2}} + C}$$

$$\boxed{A = 30(100+2t) + C(100+2t)^{-\frac{1}{2}}}$$

@ $t=0$ $\quad 500L\cdot\frac{2g}{L} = 30(100) + C(100)^{-\frac{1}{2}}$

$$1000 = 3000 + \frac{C}{10}$$

$$\boxed{C = -20000} \quad 2$$

OVERFLOW
@ $500+10t=1000$
$t=50$

$$A(t) = 30(100+2t) - 20000(100+2t)^{-\frac{1}{2}}$$

$$\boxed{A(50) = 30(200) - 20000(200)^{-\frac{1}{2}}}$$

$$= 6000 - \frac{2000\cancel{0}}{1\cancel{0}\sqrt{2}}$$

$$= \boxed{6000 - 1000\sqrt{2}}\ g \quad 2$$